HW of logic, set theory and functions

Please return the exercises in the order or each one on a separate sheet (not everything together in a messy mix). Do not waste time on bonus : they are interesting questions, but also more difficult.

The marking scheme is informative but not definitive. It is not necessary to do everything to get a good grade. Since the HW is pretty long, the marking could be non linear.

NB: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$.

Exercise 1 : Propositional logic (20 points)

1. Show that
$$(\varphi \Rightarrow \bot) \equiv (\neg\varphi)$$
Do not hesitate to comment.

2. Show the PIERCE's law:
$$\Big(\big((\varphi \Rightarrow \psi) \Rightarrow \varphi\big) \Rightarrow \varphi\Big)$$

3. Show
$$\big(A \Rightarrow (B \Rightarrow C)\big) \equiv \big((A \wedge B) \Rightarrow C\big)$$

4. Show that $\varphi \equiv \psi$ if and only if $\big(\varphi \Leftrightarrow \psi\big)$ is a tautology.

5. Let φ be the formula
$$\Big(\big((P \Rightarrow (\neg Q)) \Rightarrow (\neg P)\big) \wedge R\Big)$$
   (a) Write φ as a tree
   (b) Give the truth table of φ
   (c) Is this formula a tautology? Explain.
   (d) Is this formula a contradiction? Explain.

6. (a) Prove BOOLE's laws:

$$\big((A \vee B) \wedge C\big) \equiv \big((A \wedge C) \vee (B \wedge C)\big)$$
$$\big((A \wedge B) \vee C\big) \equiv \big((A \vee C) \wedge (B \vee C)\big)$$

   That's the distributivity of $\vee$ with respect to $\wedge$ and the distributivity of $\wedge$ wrt. $\vee$.

   (b) Show that $(A \wedge B) \equiv (B \wedge A)$ and $(A \vee B) \equiv (B \vee A)$. That's the commutativity of $\vee$ and $\wedge$

   (c) Show that $\big((A \wedge B) \wedge C\big) \equiv \big(A \wedge (B \wedge C)\big)$ and $\big((A \vee B) \vee C\big) \equiv \big(A \vee (B \vee C)\big)$. That's the associativity of $\vee$ and $\wedge$.

7. Which integers of the interval $[\![2, 40]\!]$ satisfy the implication

$$\big(n \text{ is a multiple of } 10 \Rightarrow (n+1) \text{ is a prime number}\big)$$

Exercise 2 : Functions (15 points)

1. Let

$$f : \mathbb{C} \setminus \{-3\} \to \mathbb{C}$$
$$z \mapsto \frac{iz - i}{z + 3}$$

   (a) Show that f is injective.

   (b) Determine E such that

$$g : \mathbb{C} \setminus \{-3\} \to E$$
$$z \mapsto f(z)$$

   is bijective.

   (c) Find the inverse bijection of g.

2. Let $f : \mathbb{R} \to \mathbb{R}$. We assume f is increasing. Show that f is injective.

3. (5 points bonus)

   (a) Show that there exists a bijection between $]0, 1[$ and $\mathbb{R}$.

   (b) Is there a bijection between $[0; 1]$ and $]0; 1[$? Explain.

Exercise 3 : Relations (15 points)

1. We define the relation $\mathscr{R}$ on $\mathbb{R}$ by $x \mathscr{R} y$ if and only if $x^2 - y^2 = x - y$.
   (a) Show that $\mathscr{R}$ is a equivalence relation.
   (b) Let $x \in \mathbb{R}$. Find the equivalence class of x.
2. We define on $\mathbb{N}^2$ the relation $\sqsubseteq$ by

$$(x, y) \sqsubseteq (x', y') \Leftrightarrow \Big(x < x' \vee \big(x = x' \wedge y \leqslant y' \big) \Big)$$

   (a) Show that $\sqsubseteq$ is an order relation.
   (b) (5 points bonus) Show that this order is well founded
   (c) (5 points bonus) Is this relation still a relation order on $\mathbb{Z}^2$? Is it a well founded order?

Exercise 4 : Set theory (15points)

1. Draw the following subset of $\mathbb{R}^2$
   (a) $\complement \left([-1, 1] \times [-1, 1] \right)$
   (b) $\left(\complement [-1, 1] \right) \times \left(\complement [-1, 1] \right)$
   (c) $\left\{ (a, b) \in \mathbb{Z}^2 \;\middle|\; a^2 + b^2 \leqslant 4 \right\}$
2. Write formally each of these three subsets of $\mathbb{R}^2$. The "cross" is only one set and it is infinite in the four directions (here cut by the frame of the picture).

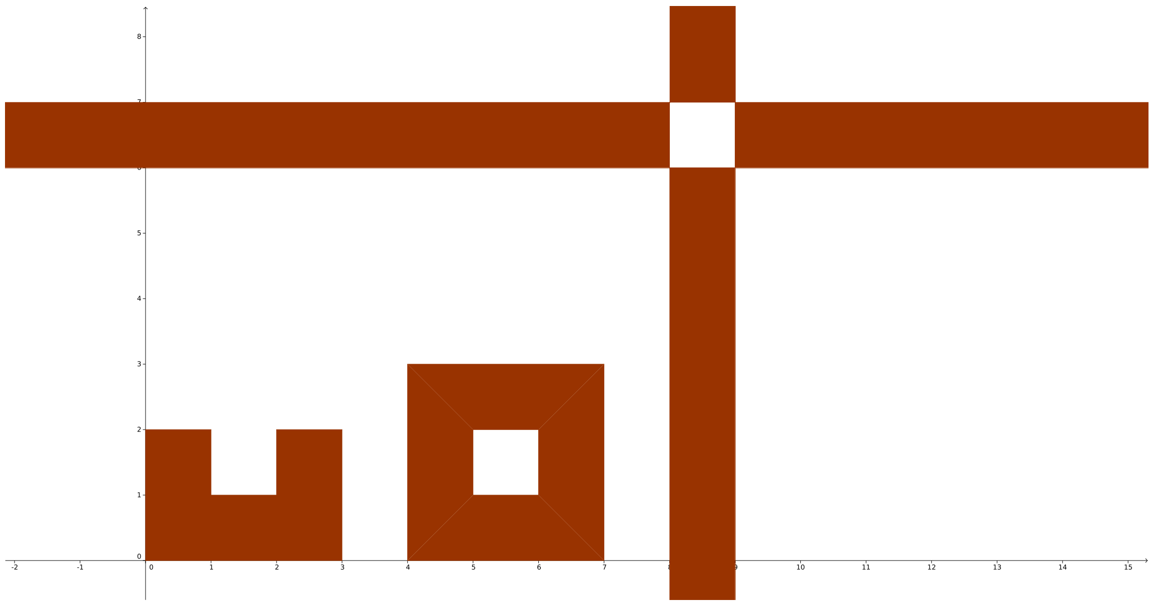

Figure 1: The subsets

3. Let E be a set. We define the indicator function of $A \subseteq E$ the application:

$$\varphi_A : E \to \{0,1\}$$
$$x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

We note that all function defined on E is an indicator function of a subset of E if and only if f maps to the set $\{0,1\}$. If f is an indicator function, it describes the set $\{x \in E \mid f(x) = 1\}$.

Let A and B be two subsets of E. Among the following functions, find which are indicator functions, and in this case, find the described subset.

(a) $1 - \varphi_A$

(b) $\min(\varphi_A, \varphi_B)$

(c) $\max(\varphi_A, \varphi_B)$

(d) $\varphi_A \varphi_B$

(e) $\varphi_A + \varphi_B$

(f) $\varphi_A - \varphi_B$

(g) (2 points bonus) $\varphi_A + \varphi_B - \varphi_A \varphi_B$

(h) (2 points bonus) $(\varphi_A - \varphi_B)^2$

4. (5 points bonus) Express $\exists! x \in E : P(x)$ using only the quantifiers $\exists$ and $\forall$.

Exercise 5 : Induction (15 points)

1. Show that $\forall n \geqslant 4, n < 2^n < n!$. Where $n! = 1 \times 2 \times \cdots \times n$ (NB : $0! = 1$). One could use, without proof, that $(n+1)! = (n+1) \cdot n!$.

2. Show that for all natural numbers n (ie. non negative integers)

$$\forall x \in \mathbb{R}^{+*}, (1+x)^n \geqslant 1 + nx$$

3. (5 points bonus) Show that for all $n \in \mathbb{N}^*, n(2n+1)(7n+1)$ is divisible by 6.