TD3 : Functions

Exercise 1:

(a) Are the following functions injective, surjective, bijective ?

i.
$$f_1 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto 2n$$

ii.
$$f_2 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto -n$$

iii.
$$f_3 : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$

iv.
$$f_4 : \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto x^2$$

v.
$$f_5 : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto x^2$$

(b) Are the following functions injective, surjective, bijective ?

i.
$$f_1 : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto n+1$$

ii.
$$f_1 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto n+1$$

iii.
$$f_3 : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (x+y, x-y)$$

Exercise 2:

Let f and g be two mappings from $\mathbb{N}$ to $\mathbb{N}$ defined by

$$f : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto 2n$$

$$g : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto \begin{cases} \frac{n}{2} & \text{si } x \text{ est pair} \\ 0 & \text{sinon} \end{cases}$$

Determine whether f, g, $f \circ g$ et $g \circ f$ are injective, surjective or bijective.

Exercise 3:

Show that f defined by

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^{+*} \\ x &\mapsto \frac{e^x + 2}{e^{-x}} \end{aligned}$$

is bijective. Compute the inverse bijection. We could use the substitution $X = e^x$.

Exercise 4:

(a) Let f

$$\begin{aligned} f : \mathbb{N} &\to \mathfrak{P} \\ n &\mapsto 2n \end{aligned}$$

where $\mathfrak{P}$ is the set of even natural numbers Let g

$$\begin{aligned} g : \mathbb{Z}^{-*} &\to \mathfrak{I} \\ n &\mapsto -2n + 3 \end{aligned}$$

where $\mathfrak{I}$ is the set of odd natural numbers. Show that f and g are bijections.

(b) We let h

$$\begin{aligned} h : \mathbb{Z} &\to \mathbb{N} \\ n &\mapsto \begin{cases} f(n) & \text{si } n \geqslant 0 \\ g(n) & \text{sinon} \end{cases} \end{aligned}$$

Show that h is a bijection.

Exercise 5:

Let

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{C} \\ t &\mapsto e^{it} \end{aligned}$$

Find subsets of $\mathbb{R}$ and $\mathbb{C}$ such that f is a bijection

Exercise 6:

Let

$$\begin{aligned} f : [1, +\infty[&\to [0, +\infty[\\ x &\mapsto x^2 - 1 \end{aligned}$$

Determine whether f is injective, surjective, bijective...

Exercise 7: Harder curiosities

(a) Find a bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

(b) Find a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.