TD3 : Functions

Exercise 1:

(a) Are the following functions injective, surjective, bijective ?

i.

$$f_1 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto 2n$$

Solution:
- Injectivity : Let $(x, y) \in \mathbb{Z}^2$. We assume $f_1(x) = f_1(x)$. On a donc $2x = 2y$. So $x = y$. Thus f_1 is injective.
- Surjectivity : We note for all$n \in \mathbb{Z}$, $f_1(n)$ is even. Thus, the equation $f_1(n) = 1$ has no solution. So f_1 is not surjective
- Bijectivity : Since f_1 is not surjective, it is not bijective.

ii.

$$f_2 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto -n$$

Solution:
- Injectivity : Let $(x, y) \in \mathbb{Z}^2$. We assume $f_2(x) = f_2(x)$. thus $-x = -y$ and $x = y$. So f_2 is injective.
- Surjectivity : Let $n \in \mathbb{Z}$. We are trying to solve $f_2(m) = n$ for m. That is $-m = n$. So $m = -n$. This solution exists. So the function in surjective
- Bijectivity : f_2 is injective and surjective, so bijective. On the other hand, we could have been satisfied by noting $f_2(m) = n$ has a unique solution for m.

iii.

$$f_3 : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$

Solution:
- Injectivity : $f_3(-1) = 1 = f_3(1)$ so f_3 is not injective.
- Surjectivity : The equation $f_3(x) = -1$ has no solution. So f_3 si not surjective.
- Bijectivity : f_3 is neither injective, nor surjective, so no way it is bijective.

iv.

$$f_4 : \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto x^2$$

Solution:

- Injectivity : See above.
- Surjectivity : All elements of $\mathbb{R}^+$ are reached. More formally, let $y \in \mathbb{R}^+$. We look for the solution for x of $f_4(x) = y$. A solution (non unique) is $x = \sqrt{y}$ (which exists). So f_4 is surjective.
- Bijectivity : The function i s not injective so not bijective.

v.

$$f_5 : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto x^2$$

Solution:

- Injectivity : See above.
- Surjectivity : Let $y \in \mathbb{C}$. y can be written in exponential form : $y = \rho e^{i\theta}$. We define $x = \sqrt{\rho} e^{i\frac{\theta}{2}}$. We have $f_5(x) = y$. So, for all $y \in \mathbb{C}$, the equation $f_5(x) = y$ has a solution for x, so f_5 is surjective.
- Bijectivity : The function is not injective, so not bijective.

(b) Are the following functions injective, surjective, bijective ?

i.

$$f_1 : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto n + 1$$

Solution:

- Injectivity : Let $(x, y) \in \mathbb{N}^2$. We assume $f_1(x) = f_1(x)$. We have $x + 1 = y + 1$. So $x = y$. So f_1 is injective.
- Surjectivity : $f_1(x) = 0$ has no solution, so f_1 is not surjective.
- Bijectivity : Since f_1 is not surjective, it is not bijective.

ii.

$$f_1 : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto n + 1$$

Solution:

- Injectivity : Let $(x, y) \in \mathbb{Z}^2$. We assume $f_2(x) = f_2(x)$. We have $x + 1 = y + 1$. So $x = y$. So f_1 is injective.
- Surjectivity : Let $y \in \mathbb{Z}$. We look for a solution to $f_2(x) = y$. A solution is $x = y - 1$. so f_2 is surjective.
- Bijectivity : Since f_2 is injective and surjective, f_2 is bijective.

iii.

$$f_3 : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + y, x - y)$$

Solution:

- Injectivity : Let $((a, b), (c, d)) \in \mathbb{R}^2 \times \mathbb{R}^2$. We assume $f_3((a, b)) = f_3((c, d))$. Namely $(a + b, a - b) = (c + d, c - d)$, so $a + b = c + d$ et $a - b = c - d$. We solve

$$\begin{cases} a + b = c + d \\ a - b = c - d \end{cases}$$

That is

$$\begin{cases} a + b = c + d \\ 2a = 2c \end{cases}$$

That is

$$\begin{cases} b = d \\ a = c \end{cases}$$

So $(a, b) = (c, d)$, so f_3 is injective.

- Surjectivity : Let $(c, d) \in \mathbb{R}^2$. We solve $f_3((a, b)) = (c, d)$ for (a, b).

$$\begin{cases} a + b = c \\ a - b = d \end{cases}$$

That is

$$\begin{cases} a + b = c \\ 2a = c + d \end{cases}$$

consequently

$$\begin{cases} b = \frac{1}{2}c - \frac{1}{2}d \\ a = \frac{1}{2}c + \frac{1}{2}d \end{cases}$$

So there is at least a solution, so f_3 is surjective.

- Bijectivity : f_3 is injective and surjective, so bijective.

Exercise 2:

Let f and g be two mappings from $\mathbb{N}$ to $\mathbb{N}$ defined by

$$f : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto 2n$$

$$g : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto \begin{cases} \frac{n}{2} & \text{si } x \text{ est pair} \\ 0 & \text{sinon} \end{cases}$$

Determine whether f, g, $f \circ g$ et $g \circ f$ are injective, surjective or bijective.

Solution:

- f is obviously injective. f never takes the value 1, so f is not bijective.
- g is not injective because $g(1) = 0 = g(3)$. g is surjective since all $y \in \mathbb{N}$ is reached, indeed $g(2y) = y$. g is not bijective.
- Let $x \in \mathbb{N}$. $g \circ f(x) = g(2x)$. Yet $2x$ is even, thus $g(2x) = x$. So $g \circ f = Id$. So $g \circ f$ is bijective.
- We have $f \circ g(0) = 0 = f \circ g(1)$ so $f \circ g$ is not injective. We also have $f \circ g(x) = 2g(x)$. So $f \circ g(x)$ is always even, so $f \circ g$ is not surjective. So $f \circ g$ is not bijective.

Exercise 3:

Show that f defined by

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^{+*} \\ x &\mapsto \frac{e^x + 2}{e^{-x}} \end{aligned}$$

is bijective. Compute the inverse bijection. We could use the substitution $X = e^x$.

Solution: Let $y \in \mathbb{R}^{+*}$. We want to solve $f(x) = y$ for x. So we want to solve $\frac{e^x+2}{e^{-x}} = y$. Using the suggested substitution : $\frac{X+2}{\frac{1}{X}} = y$.

$$\begin{aligned} \tfrac{X+2}{\frac{1}{X}} = y \quad &\Leftrightarrow \quad X(X+2) = y \\ &\Leftrightarrow \quad X^2 + 2X = y \\ &\Leftrightarrow \quad X^2 + 2X - y = 0 \end{aligned}$$

$\Delta = 4 - 4 \cdot (-y) = 4(1+y)$. The solutions are

$$X = -1 \pm \sqrt{1+y}$$

But $X = e^x$. So $X > 0$. So $e^x = X = -1 + \sqrt{1+y}$. We infer that $x = \ln\left(-1 + \sqrt{1+y}\right)$. There is at most one solution. There is still to check that $\ln\left(-1 + \sqrt{1+y}\right)$ is well defined. We have $y > 0$, so $\sqrt{1+y} > 1$, so $-1 + \sqrt{1+y} > 0$, so the solution is always well defined. So the function isbijective, and its inverse bijection is

$$\begin{aligned} f^{-1} : \mathbb{R}^{+*} &\to \mathbb{R} \\ y &\mapsto \ln\left(-1 + \sqrt{1+y}\right) \end{aligned}$$

Exercise 4:

(a) Let f

$$\begin{aligned} f : \mathbb{N} &\to \mathfrak{P} \\ n &\mapsto 2n \end{aligned}$$

where $\mathfrak{P}$ is the set of even natural numbers Let g

$$\begin{aligned} g : \mathbb{Z}^{-*} &\to \mathfrak{I} \\ n &\mapsto -2n+3 \end{aligned}$$

where $\mathfrak{I}$ is the set of odd natural numbers. Show that f and g are bijections.

Solution:

- Let $(x, y) \in \mathbb{N}^2$. We assume $f(x) = f(y)$ so $2x = 2y$ so $x = y$, so f is injective. On the other hand, for all even number n, there is a solution for m to $f(m) = n$: $\frac{n}{2} = m$. So f is surjective, so bijective.
- Let $(x, y) \in \mathbb{Z}^{-*2}$. We assume that $g(x) = g(y)$ so $-2x+3 = -2y+3$ so $x = y$, so g is injective.
  Let $n = 2k+1$ be a odd number. We want to solve $g(x) = n$ for x. $-2n+3 = 2k+1$, $-2n+2 = 2k$, so $-n+1 = k$ so $n-1 = -k$, so $n = 1-k$. so g is surjective so bijective.

(b) We let h

$$\begin{aligned} h : \mathbb{Z} &\to \mathbb{N} \\ n &\mapsto \begin{cases} f(n) & \text{si } n \geqslant 0 \\ g(n) & \text{sinon} \end{cases} \end{aligned}$$

Show that h is a bijection.

Solution:

- Injectivity : Let $(x, y) \in \mathbb{Z}^2$. We assume $h(x) = h(y)$. We distinguish 3 cases :
  - x and y are non negative. We thus have $f(x) = h(x) = h(y) = f(y)$. So $x = y$ thank to the injectivity of f.
  - x and y are negative. We have $g(x) = h(x) = h(y) = g(y)$. So $x = y$ thank to the injectivity of g.
  - x and y have opposite signs. Thus $h(x)$ is even and $h(y)$ is odd or conversely. So we cannot have $h(x) = h(y)$. This case is so impossible.

  In all these cases, h is injective.
- Surjectivity : Let $y \in \mathbb{N}$. We distinguish 2 cases :
  - y is even : there is $x \in \mathbb{N}$ such that $f(x) = y$, by surjectivity of f. So $h(x) = y$ has a solution.
  - y is odd : there is $x \in \mathbb{Z}^{-*}$ such that $g(x) = y$, by injectivity of g. So $h(x) = y$ has a solution.

  Thus h is surjective.
- So h is bijective.

Exercise 5:

Let

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{C} \\ t &\mapsto e^{it} \end{aligned}$$

Find subsets of $\mathbb{R}$ and $\mathbb{C}$ such that f is a bijection

> **Solution:** The function g
>
> $$\begin{aligned} g : [0;2\pi[&\to \mathbb{U} \\ t &\mapsto e^{it} \end{aligned}$$
>
> is bijective where $\mathbb{U}$ is the unit circle.
> g is injective :
>
> $$\begin{aligned} g(x) = g(y) &\Leftrightarrow e^{ix} = e^{iy} \\ &\Leftrightarrow x = y + 2k\pi \\ &\Leftrightarrow x = y \text{ since } t, t' \in [0, 2\pi[\text{ and so } k = 0 \end{aligned}$$
>
> g is surjective since all complex number of $\mathbb{U}$ can be written under the form $e^{i\theta}$, and we can choose $\theta \in [0, 2\pi[$.

Exercise 6:

Let

$$\begin{aligned} f : [1, +\infty[&\to [0, +\infty[\\ x &\mapsto x^2 - 1 \end{aligned}$$

Determine whether f is injective, surjective, bijective...

> **Solution:**
>
> — Let $(x, y) \in [1, +\infty[^2$. We assume $f(x) = f(y)$. So $x^2 - 1 = y^2 - 1$ so $x^2 = y^2$ and $x = \pm y$. But $x, y > 1$. And $x = y$. So f is injective.
>
> — Let $y \in [0, +\infty[$. We want to solve $f(x) = y$ for x in $[1, +\infty[$. We have $x^2 - 1 = y$, $x^2 = y + 1$ so $x = \pm\sqrt{y+1}$. But $x > 0$. So $x = \sqrt{y+1}$. So f is surjective.
>
> — So f is injective and surjective so bijective.

Exercise 7: Harder curiosities

(a) Find a bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

> **Solution:**
>
> $$\begin{aligned} f : \mathbb{N}^2 &\to \mathbb{N} \\ (m, n) &\mapsto 2^m(2n+1) - 1 \end{aligned}$$
>
> makes the job done. Indeed we can show that
>
> $$\begin{aligned} g : \mathbb{N}^2 &\to \mathbb{N}^* \\ (m, n) &\mapsto 2^m(2n+1) \end{aligned}$$
>
> is a bijection.
>
> — Let $((a, b), (c, d) \in \mathbb{N}^2 \times \mathbb{N}^2$. We assume $g((a, b)) = g((c, d))$. We thus have $2^a(2b+1) = 2^c(2d+1)$. 2^a et 2^c are even, and $2b+1$ and $2d+1$ are odd. We have $2^a = 2^c$ and $2b+1 = 2d+1$. So $(a, b) = (c, d)$. So g is injective.

— Let $y \in \mathbb{N}$. Let 2^k the bigger power of 2 that divides y. There exists l such that $y = 2^k l$ with l odd. Indeed, if l is even, we could divide y by 2^{k+1}, which contradict the choice of k. Since l is odd, there is n such that $l = 2n + 1$. So $x = (k, n)$ is a solution of $g(x) = y$. We infer that g is surjective.

— g is bijective.

We infer easily that f is bijective, too.

(b) Find a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.

Solution: So many steps!

— Show that $\mathcal{P}(\mathbb{N})$ is in bijection with the set of indicator function of $\mathbb{N}$

— Show that these functions are in bijection with the infinite sequences that map in the set $\{0, 1\}$.

— Show that these sequences are in bijection with the binary decompositions of number of $[0, 1]$. For that, first, build a surjection, determine the elements reached multiple times (exactly 2 actually). Build a sequence by listings the preimages of these values and skipping one value in two.

— Build a bijection between $[0, 1]$ and $]0, 1[$. Build a bijection between $]0, 1[$ and $\mathbb{R}$.

— Pfiou finished!