Math exam

1 Propositional logic

Let ψ be the formula $\big((A \vee (\neg B)) \Rightarrow B\big)$.

1. Write the truth table of ψ.

Solution:

$[A]_\sigma$	$[B]_\sigma$	$[(\neg B)]_\sigma$	$\big[(A \vee (\neg B))\big]_\sigma$	$[\psi]_\sigma$
ff	ff	tt	tt	ff
ff	tt	ff	ff	tt
tt	ff	tt	tt	ff
tt	tt	ff	tt	tt

2. Is ψ a tautology?

Solution: No: the environment

$$A \mapsto ff$$
$$B \mapsto ff$$

makes the formula evaluate to ff.

3. Is ψ a contradiction?

Solution: No: the environment

$$\begin{aligned} A &\mapsto \mathit{ff} \\ B &\mapsto \mathit{tt} \end{aligned}$$

makes the formula evaluate to tt.

2 Induction

Let

$$\begin{aligned} f : \mathbb{R}^+ &\to \mathbb{R} \\ x &\mapsto \frac{x}{x+1} \end{aligned}$$

For $n \in \mathbb{N}^*$, we denote

$$f^{(n)}(x) = \begin{cases} f(x) & \text{if } n = 1 \\ f \circ f^{(n-1)}(x) & \text{otherwise} \end{cases}$$

1. Show that the predicate P_n : "$\forall x \in \mathbb{R}^+, f^{(n)}(x) = \frac{x}{nx+1}$" holds over $\mathbb{N}^*$

Solution: Let's prove P_n by induction on $\mathbb{N}^*$.

- Base. For $n = 1$. Let $x \in \mathbb{R}^+$. We have $f^{(1)}(x) = f(x) = \frac{x}{x+1}$. Hence P_1.
- Induction. Let $n \in \mathbb{N}^*$. We assume that P_n is true. Let's prove P_{n+1}.

  We have $\forall x \in \mathbb{R}^+, f^{(n)}(x) = \frac{x}{nx+1}$. Since $x \geqslant 0$, $f^{(n)}(x) \geqslant 0$. f is well-defined over $\mathbb{R}^+$. Thus $f \circ f^{(n)}$ is well-defined over $\mathbb{R}^+$.

  Let $x \in \mathbb{R}^+$

  $$\begin{aligned} f^{(n+1)}(x) &= f \circ f^{(n)}(x) \\ &= \frac{\frac{x}{nx+1}}{\frac{x}{nx+1} + 1} \\ &= \frac{\frac{x}{nx+1}}{\frac{x+nx+1}{nx+1}} = \frac{x}{(n+1)x+1} \end{aligned}$$

Hence P_{n+1}

So, we have $\forall n \in \mathbb{N}^*, P_n$.

3 Relations

Let $\mathcal{R}$ be the relation defined over $\mathbb{R}^2$ by

$$(a,b) \,\mathcal{R}\, (c,d) \text{ ssi } a+d = c+b$$

1. Show that $\mathcal{R}$ is an equivalence relation over $\mathbb{R}^2$.

Solution:

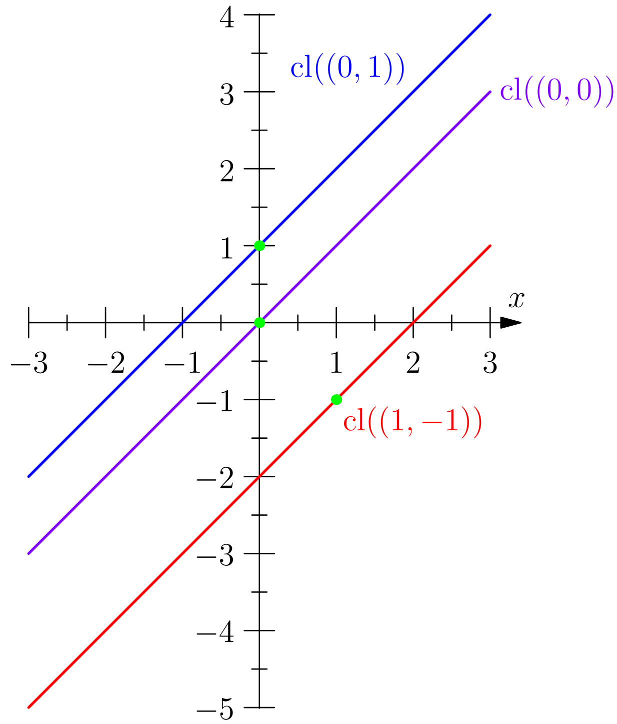
- Reflexivity. Let $(x,y) \in \mathbb{R}^2$. We have $x+y = y+x$. Thus $(x,y) \,\mathcal{R}\, (x,y)$.
- Symmetry. Let $(x,y),(x',y') \in \mathbb{R}^2$. We assume $(x,y) \,\mathcal{R}\, (x',y')$. It comes $x+y' = x'+y$, so $x'+y = y'+x$. And $(x',y') \,\mathcal{R}\, (x,y)$.
- Transitivity. Let $(x,y),(x',y'),(x'',y'') \in \mathbb{R}^2$. We assume$(x,y) \,\mathcal{R}\, (x',y')$ and $(x',y') \,\mathcal{R}\, (x'',y'')$. We have $x+y' = y+x'$ et $x'+y'' = y'+x''$. By a member-wise sum: $x+y'+x'+y'' = y+x'+y'+x''$, thus $x+y'' = y+x''$. So $(x,y) \,\mathcal{R}\, (x'',y'')$.

2. Find the equivalence classes of $(0,1)$, $(0,0)$ and $(1,-1)$. Draw these sets in the $\mathbb{R}^2$ plane.

Solution:

- Let $(x,y) \in \mathbb{R}^2$. We assume $(x,y) \,\mathcal{R}\, (0,1)$. We have $x+1 = y+0$. It comes $\operatorname{cl}\big((0,1)\big) = \big\{(x,x+1) \mid x \in \mathbb{R}\big\}$.
- Let $(x,y) \in \mathbb{R}^2$. We assume $(x,y) \,\mathcal{R}\, (0,0)$. We have $x+0 = y+0$. It comes $\operatorname{cl}\big((0,0)\big) = \big\{(x,x) \mid x \in \mathbb{R}\big\}$.

- Let $(x, y) \in \mathbb{R}^2$. We assume $(x,y) \mathcal{R} (1,-1)$. We have $x - 1 = y + 1$. It comes $\operatorname{cl}\big((1,-1)\big) = \big\{(x, x-2) \,\big|\, x \in \mathbb{R}\big\}$.



4 Functions

1. Let $c \in \mathbb{R}$. Solve for $X \in \mathbb{R}^{+*}$ the equation

$$X^2 - 2cX - 1 = 0$$

Solution: $\Delta = 4c^2 + 4 > 0$. We thus have two solutions:

$$X_1 = c - \sqrt{c^2+1} \text{ et } X_2 = c + \sqrt{c^2+1}$$

However, $|c| < git\sqrt{c^2+1}$. So $X_1 < 0$ and $X_2 > 0$. Thus, the only solution is
$$X_2 = c + \sqrt{c^2+1}$$

2. Solve over $\mathbb{R}$ the equation parametrized by $c \in \mathbb{R}$
$$\frac{e^x - e^{-x}}{2} = c$$
We could let $X = e^x$.

Solution: By letting $X = e^x$, we have $X > 0$ and the equation become
$$\begin{aligned}\frac{X - \frac{1}{X}}{2} = c &\Leftrightarrow \frac{\frac{X^2-1}{X}}{2} = c \\ &\Leftrightarrow \frac{X^2-1}{2X} = c \\ &\Leftrightarrow X^2 - 1 = 2Xc \\ &\Leftrightarrow X^2 - 2Xc - 1 = 0\end{aligned}$$
By referring to the previous question, we find $X = c + \sqrt{c^2+1}$ hence $x = \ln\left(c + \sqrt{c^2+1}\right)$, which is well-defined since $X > 0$.

3. Let sinh be the hyperbolic sine function defined by
$$\begin{aligned}\sinh : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{e^x - e^{-x}}{2}\end{aligned}$$
Is it injective, surjective and/or bijective ?

Solution: Since the equation $\sinh(x) = c$ has a solution for all $c \in \mathbb{R}$, sinh is surjective.

Moreover there is exactly one solution, sinh si injective.

Thus, sinh is bijective.