Math exam
Linear algebra

Allons, courage et confiance !

1 Linear subspaces?

Everything must come with a justification.

1. Let $T \in \mathbb{R}^*$. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is T-periodic if $\forall x \in \mathbb{R}, f(x+T) = f(x)$. Is the set of T-periodic function a linear subspace of the linear space of function from $\mathbb{R}$ to $\mathbb{R}$?

Solution: Yes, clearly. Let $\lambda \in \mathbb{R}$, and f and g be functions of $\mathbb{R} \to \mathbb{R}$.

Let $x \in \mathbb{R}$,

$$\begin{aligned}(\lambda f + g)(x+T) &= \lambda f(x+T) + g(x+T) \\ &= \lambda f(x) + g(x) \\ &= (\lambda f + g)(x)\end{aligned}$$

Thus $\lambda f + g$ is T-periodic.

In the other hand, the set of periodic functions is not a linear subspace: $x \mapsto \sin(2\pi x)$ and $x \mapsto \sin\left(\frac{2\pi}{\sqrt{2}}x\right)$ are both periodic, but their sum isn't! Indeed, the former is 1-periodic, while the second is $\sqrt{2}$-periodic. If the sum is T-periodic, then T is a multiple of 1, thus it is an integer. But it is also a multiple of $\sqrt{2}$. Then, $\sqrt{2}$ should be rational. We know it is not true. Tada !

I smuggle discretely a theorem on additive subgroups of $(\mathbb{R}, +)$: they are either dense in $\mathbb{R}$, or of the form $a\mathbb{Z}$ where $a \in \mathbb{R}$. Periods of a function is an additive subgroup of $\mathbb{R}$, and it cannot be dense except if the function is constant (when it is continuous).

2. Is $\left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leqslant 0\right\}$ a linear subspace of $\mathbb{R}^2$?

Solution: Yes! $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leqslant 0\} = \{0\}$. It is a linear subspace.

3. Is $\{(x,y,z) \in \mathbb{R}^3 \mid x+y=0 \vee x+z=0\}$ a linear subspace of $\mathbb{R}^3$?

Solution: No! A disjunction of conditions, that stinks. More precisely, if we take $(1,-1,0)$ and $(1,0,-1)$, both are in the subset. The sum is $(2,-1,-1)$, which is not in the subset.

4. Is the set of divergent sequences a linear subspace of the linear space of real-valued sequences?

Solution: No. The null sequence does not diverge, thus we are screwed: a linear subspace always includes 0.

5. Let $(a,b) \in \mathbb{R}^2$. Is the set of real-valued sequences $(u_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\forall i \in \mathbb{N}, au_i + bu_{i+1} = u_{i+2}$ a linear subspace of the linear space of real-valued sequences?

Solution: Yes!

Let $(u_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be two such sequences, and $\lambda \in \mathbb{R}$. Let $i \in \mathbb{N}$

$$\begin{aligned} a(\lambda u_i + v_i) + b(\lambda u_{i+1} + v_{i+1}) - (\lambda u_{i+2} + v_{i+2}) &= \lambda\left(au_i + bu_{i+1} - u_{i+2}\right) + av_i + bv_{i+1} - v_{i+2} \\ &= \lambda 0 + 0 \\ &= 0 \end{aligned}$$

Thus $\lambda u + v$ match also the linear relation. Victory!