Linear Algebra Quiz

1 Without computation

Toute cette section peut se faire sans calcul, ou très très succinct, le genre de calcul qu'on n'écrit même pas.

All this section can be solved without computation, or very very short ones. The kind of computations we do not even write.

1. We work in the vector space $\mathbb{R}^3$. Is the family made of these two vectors

$$(1, 1, 7)$$
$$(\pi, \pi^2, \pi^3)$$

   a spanning set?

   Solution: No. $\mathbb{R}^3$ has dimension 3 and this family has 2 vectors. It cannot span $\mathbb{R}^3$.

2. Still in $\mathbb{R}^3$, justify that $F = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x = 0 \wedge 7y - 5z = 0\right\}$ is a linear subspace of $\mathbb{R}^3$.

   Solution: F is the kernel of the linear map $(x, y, z) \mapsto (x, 7y - 5z)$. Yet a kernel is always a linear subspace.

3. Show that

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 7 \\ 0 & 0 & 12 \end{pmatrix}$$

   is not invertible.

   Solution: This matrix has a null column, thus, the family of column vectors is linearly dependent.

2 Computations

Do not forget to justify (at least a bit) the result, either by detailing the computation, either by citing the method or by checking the result *a posteriori* (if applicable). Writing computations steps are not really necessary, but advised. Especially if the result is wrong.

1. Find the inverse of

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Solution: We solve

$$\begin{cases} x_1 + 2x_2 = y_1 \\ 2x_1 + x_2 = y_2 \end{cases}$$

we immediately have $3x_2 = 2y_1 - y_2$ and $-3x_1 = y_1 - 2y_2$. Thus, the inverse is

$$\frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

2. (a) We are given

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Show that $A = D + N$.

Solution: Obvious, isn't it?

(b) Show that $DN = ND$.

Solution: Just boring computations...

$$DN = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}$$

$$ND = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}$$

(c) Compute N^2 and N^3, and deduce the value of N^k for $k \geqslant 2$

Solution: We have

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can directly infer that $\forall n \geqslant 2, N^2 = 0$.

(d) Let $k \in \mathbb{N}$. Compute D^k.

Solution: We have

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 0 & 3^k \end{pmatrix}$$

(e) Let $k \in \mathbb{N}$. Compute A^k

Solution: We have $A^k = (D+N)^k = \sum_{n=0}^{k} D^{n-k} N^k$. Yet, for $k \geqslant 2$, $N^k = 0$. Thus $A^k = D^n N^0 + D^{n-1} N = 3^n I_3 + 3^{n-1} N = 3^{n-1}(3I3 + N) = 3^{n-1} A$.