Math exam

1 Propositional logic

We are going to take a look at a class of logic formulæ that are very useful in logic programming and formal methods: HORN clauses.

Given a family of logic variables $(A_i)_{i\in\mathbb{N}}$, we let:

$$\bigwedge_{i=0}^{n} A_i = A_0 \wedge A_1 \wedge \cdots \wedge A_n = A_n \wedge \bigwedge_{i=0}^{n-1} A_i$$

and the dual

$$\bigvee_{i=0}^{n} A_i = A_0 \vee A_1 \vee \cdots \vee A_n = A_n \vee \bigvee_{i=0}^{n-1} A_i$$

We can see it looks like the notation $\sum\limits_{i=0}^{n}$, except that we use $\wedge$ or $\vee$ instead of $+$.

Given a logic variable B and a family $(A_i)_{i\in\mathbb{N}}$, a HORN clause is a formula of the form

$$\left(\bigwedge_{i=0}^{n} A_i\right) \Rightarrow B$$

where $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we let H_n be the formula $\left(\bigwedge\limits_{i=0}^{n} A_i\right) \Rightarrow B$.

Here are some examples of HORN clauses:

- H_0: $A_0 \Rightarrow B$
- H_1: $(A_0 \wedge A_1) \Rightarrow B$
- H_2: $(A_0 \wedge A_1 \wedge A_2) \Rightarrow B$
- ...

1. Prove that $H_0 \equiv (\neg A_0) \vee B$.
2. Prove that $H_1 \equiv (\neg A_0) \vee (\neg A_1) \vee B$.

3. Prove that $H_2 \equiv (\neg A_0) \vee (\neg A_1) \vee (\neg A_2) \vee B$. Given the number of variables, it is advised to not use a truth table.

4. More generally, we would like to prove that $\forall n \in \mathbb{N}, H_n = \left(\bigvee_{i=0}^{n} \neg A_i \right) \vee B$.

   (a) Prove that $\forall n \in \mathbb{N}, H_n \equiv \left(\neg \left(\bigwedge_{i=0}^{n} A_i \right) \right) \vee B$.

   (b) Show by induction that $\forall n \in \mathbb{N}, \neg \left(\bigwedge_{i=0}^{n} A_i \right) \equiv \bigvee_{i=0}^{n} (\neg A_i)$. We could use the expression on the very right of the defintions of $\bigwedge$ and $\bigvee$. We can see it looks like a generalization of De Morgan laws.

   (c) Infer from previous results that $\forall n \in \mathbb{N}, H_n = \left(\bigvee_{i=0}^{n} \neg A_i \right) \vee B$.

2 More logic functions

1. We define the logic function $\oplus$ (called "xor", "exclusive or", or "exclusive disjunction") by
$$A \oplus B := ((\neg A) \wedge B) \vee (A \wedge (\neg B))$$
Show that $A \oplus B \equiv \neg(A \Leftrightarrow B)$.

2. We define the logic function INH (called "inhibitor") by
$$A \text{ INH } B := A \wedge (\neg B)$$
Write the truth table of INH. I invite you to wait after the exam to think about the deep meaning of this function.

3 -jectivity

1. Let E be a set. Let $f : E \to E$. We assume that $\forall x \in E, f \circ f(x) = x$ ie. $f \circ f = Id_E$. Show that f is bijective.

2. Let A be a set. Let

$$\begin{aligned} g : \mathcal{P}(A) &\to \mathcal{P}(A) \\ X &\mapsto \complement_A X \end{aligned}$$

We recall that $\complement_A X$ is the complement of X in A, that is $A \setminus X$. Compute $g \circ g$ and deduce that g is bijective.

3. We recall that $\mathbb{Q}$ is the set of rational numbers, ie. numbers that may be written as $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$.
   (a) Let $x \in \mathbb{Q}$. Show that $1 - x \in \mathbb{Q}$.
   (b) Let

$$\begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 - x & \text{if } x \in \mathbb{Q} \\ x & \text{otherwise} \end{cases} \end{aligned}$$

   Prove that $\forall x \in \mathbb{R}, h \circ h(x) = x$ by case analysis.
   (c) Deduce that h is bijective.