Math exam

1 Propositional logic

We are going to take a look at a class of logic formulæ that are very useful in logic programming and formal methods: HORN clauses.

Given a family of logic variables $(A_i)_{i\in\mathbb{N}}$, we let:

$$\bigwedge_{i=0}^{n} A_i = A_0 \wedge A_1 \wedge \cdots \wedge A_n = A_n \wedge \bigwedge_{i=0}^{n-1} A_i$$

and the dual

$$\bigvee_{i=0}^{n} A_i = A_0 \vee A_1 \vee \cdots \vee A_n = A_n \vee \bigvee_{i=0}^{n-1} A_i$$

We can see it looks like the notation $\sum_{i=0}^{n}$, except that we use $\wedge$ or $\vee$ instead of $+$.

Given a logic variable B and a family $(A_i)_{i\in\mathbb{N}}$, a HORN clause is a formula of the form

$$\left(\bigwedge_{i=0}^{n} A_i\right) \Rightarrow B$$

where $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we let H_n be the formula $\left(\bigwedge_{i=0}^{n} A_i\right) \Rightarrow B$.

Here are some examples of HORN clauses:

- H_0: $A_0 \Rightarrow B$
- H_1: $(A_0 \wedge A_1) \Rightarrow B$
- H_2: $(A_0 \wedge A_1 \wedge A_2) \Rightarrow B$
- ...

1. Prove that $H_0 \equiv (\neg A_0) \vee B$.

Solution: It simply is implication definition.

2. Prove that $H_1 \equiv (\neg A_0) \lor (\neg A_1) \lor B$.

Solution: We have

$$\begin{aligned} H_1 &= (A_0 \land A_1) \Rightarrow B \\ &\equiv (\neg(A_0 \land A_1)) \lor B \\ &\equiv (\neg A_0) \lor (\neg A_1) \lor B \end{aligned}$$

3. Prove that $H_2 \equiv (\neg A_0) \lor (\neg A_1) \lor (\neg A_2) \lor B$. Given the number of variables, it is advised to not use a truth table.

Solution: We have

$$\begin{aligned} H_2 &= (A_0 \land A_1 \land A_2) \Rightarrow B \\ &\equiv (\neg(A_0 \land A_1 \land A_2)) \lor B \\ &\equiv (\neg A_0) \lor (\neg(A_1 \land A_2)) \lor B \\ &\equiv (\neg A_0) \lor (\neg A_1) \lor (\neg A_2) \lor B \end{aligned}$$

4. More generally, we would like to prove that $\forall n \in \mathbb{N}, H_n = \left(\bigvee_{i=0}^{n} \neg A_i \right) \lor B$.

   (a) Prove that $\forall n \in \mathbb{N}, H_n \equiv \left(\neg \left(\bigwedge_{i=0}^{n} A_i \right) \right) \lor B$.

   Solution: We have $\psi \Rightarrow \varphi \equiv (\neg \psi) \lor \varphi$. In this case, $\psi = \bigwedge_{i=0}^{n} A_i$ and $\varphi = B$.

(b) Show by induction that $\forall n \in \mathbb{N}, \neg\left(\bigwedge_{i=0}^{n} A_i\right) \equiv \bigvee_{i=0}^{n} (\neg A_i)$. We could use the expression on the very right of the defintions of $\bigwedge$ and $\bigvee$. We can see it looks like a generalization of De Morgan laws.

Solution:

- For $n = 0$, The left-hand side is $\neg\left(\bigwedge_{i=0}^{n} A_i\right) = \neg A_0$ and the right-hand side $\bigvee_{i=0}^{n} (\neg A_i) = \neg A_0$.

- Let $n \in \mathbb{N}$. Let's assume that $\neg\left(\bigwedge_{i=0}^{n} A_i\right) \equiv \bigvee_{i=0}^{n} (\neg A_i)$. We want to show that $\neg\left(\bigwedge_{i=0}^{n+1} A_i\right) \equiv \bigvee_{i=0}^{n+1} (\neg A_i)$.

$$
\begin{aligned}
\neg\left(\bigwedge_{i=0}^{n+1} A_i\right) &= \neg\left(\left(\bigwedge_{i=0}^{n} A_i\right) \wedge A_{n+1}\right) \\
&\equiv \neg\left(\bigwedge_{i=0}^{n} A_i\right) \vee \neg A_{n+1} \\
&\equiv \left(\bigvee_{i=0}^{n} \neg A_i\right) \vee \neg A_{n+1} \\
&\equiv \left(\bigvee_{i=0}^{n+1} \neg A_i\right)
\end{aligned}
$$

(c) Infer from previous results that $\forall n \in \mathbb{N}, H_n = \left(\bigvee_{i=0}^{n} \neg A_i\right) \vee B$.

Solution: Let $n \in \mathbb{N}$. $H_n \equiv \left(\neg \left(\bigwedge_{i=0}^{n} A_i\right)\right) \vee B \equiv \left(\bigvee_{i=0}^{n} (\neg A_i)\right) \vee B$.

2 More logic functions

1. We define the logic function $\oplus$ (called "xor", "exclusive or", or "exclusive disjunction") by

$$A \oplus B := ((\neg A) \wedge B) \vee (A \wedge (\neg B))$$

Show that $A \oplus B \equiv \neg(A \Leftrightarrow B)$.

Solution:

$[A]_\sigma$	$[B]_\sigma$	$[A \oplus B]_\sigma$	$[\neg(A \Leftrightarrow B)]_\sigma$
ff	*ff*	*ff*	*ff*
ff	*tt*	*tt*	*tt*
tt	*ff*	*tt*	*tt*
tt	*tt*	*ff*	*ff*

2. We define the logic function INH (called "inhibitor") by

$$A \text{ INH } B := A \wedge (\neg B)$$

Write the truth table of INH. I invite you to wait after the exam to think about the deep meaning of this function.

Solution:

$[A]_\sigma$	$[B]_\sigma$	$[\neg B]_\sigma$	$[A \text{ INH } B]_\sigma$
ff	*ff*	*tt*	*ff*
ff	*tt*	*ff*	*ff*
tt	*ff*	*tt*	*tt*
tt	*tt*	*ff*	*ff*

3 -jectivity

1. Let E be a set. Let $f : E \to E$. We assume that $\forall x \in E, f \circ f(x) = x$ ie. $f \circ f = Id_E$. Show that f is bijective.

   Solution: Id_E is bijective, thus f is injective (right) and surjective (left), so bijective.

2. Let A be a set. Let

$$\begin{aligned} g : \mathcal{P}(A) &\to \mathcal{P}(A) \\ X &\mapsto \complement_A X \end{aligned}$$

   We recall that $\complement_A X$ is the complement of X in A, that is $A \setminus X$. Compute $g \circ g$ and deduce that g is bijective.

   Solution: Let $X \in \mathcal{P}(A)$. $g \circ g(X) = g(g(X)) = \complement_A \complement_A X = X$. Thus $g \circ g = Id_{\mathcal{P}(A)}$ so g is bijective.

3. We recall that $\mathbb{Q}$ is the set of rational numbers, ie. numbers that may be written as $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$.

   (a) Let $x \in \mathbb{Q}$. Show that $1 - x \in \mathbb{Q}$.

   Solution: Let $x \in \mathbb{Q}$. There is $(p, q) \in \mathbb{Z} \times \mathbb{Z}^*$ such that $x = \frac{p}{q}$. We have

$$\begin{aligned} 1 - x &= 1 - \frac{p}{q} \\ &= \frac{q - p}{q} \in \mathbb{Q} \end{aligned}$$

   (b) Let

$$\begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 - x & \text{if } x \in \mathbb{Q} \\ x & \text{otherwise} \end{cases} \end{aligned}$$

Prove that $\forall x \in \mathbb{R}, h \circ h(x) = x$ by case analysis.

Solution: Let $x \in \mathbb{R}$. We distinguish 2 cases:

- $x \in \mathbb{Q}$. We know that $h(x) = 1 - x \in \mathbb{Q}$. So, $h(h(x)) = h(1 - x) = 1 - (1 - x) = x$.
- $x \notin \mathbb{Q}$. So $h(h(x)) = h(x) = x$.

(c) Deduce that h is bijective.

Solution: Same as before.